Mathematical modeling of aquifers by C. Choquet

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Contents

1 Groundwater issues

1.1 An aquifer

First, let's start by defining what is an aquifer. It is an underground reservoir of water which can be confined or not. In this talk, we only focus on unconfinned aquifers which means that there are some exchanges between the surface and the aquifer. For example, there can be some recharge or discharge with the surface or some contamination, we will deal with these issues in the next parts. Moreover, this type of aquifer can be separated in three zones :

- the unsaturated zone (also called the zone of aeration) which is partially filled with air and water
- the water table, which is the separation between the saturated and unsaturated zones
- the saturated zone which is the area completely filled with water

We also have to note that the deepest aquifers in France are only 8m deep which means that they are way longer than deeper. This will be important later for some approximations.

For modeling this kind of water displacement we mainly got two choices :

- fluid flow in a partially porous medium
- fluid flow in a porous medium with a free boundary

We cannot use the second approach because it avoids the unsaturated zone and we need it to study the exchanges between the surface and the aquifer.

However, there is a risk of losing the unknown with the first choice because of the hypothesis made by considering our problem as a partially porous case.

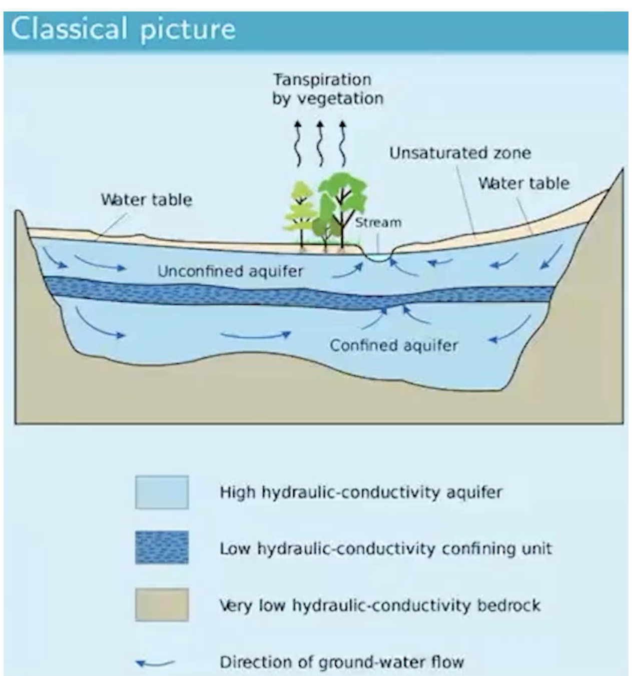

Figure 1: Sketch of an aquifer

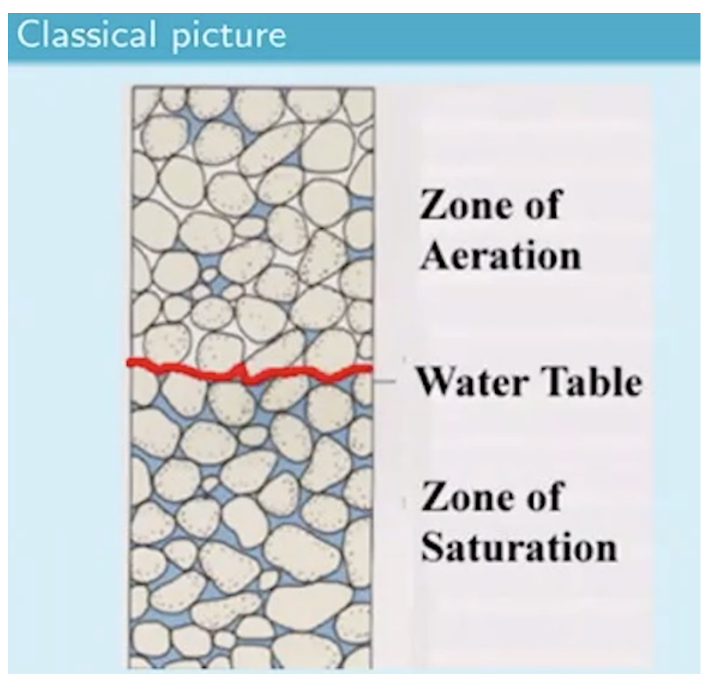

Figure 2: Two zones of an unconfined aquifer

1.2 A major societal concern

Then, we will look at the issues raised by the lack of freshwater and how modeling aquifers can be a solution.

The biggest problem with the aquifers is their vulnerability and human behavior do not help.

The three main problems evoked in the talk are the following ones :

- Withdrawing more water than what naturally comes back in the aquifer (the natural remediation)
- Contamination from human activity, for example industries or agriculture
- Natural alinization problem for coastal aquifers is worsen by the anthropic withdrawal

To develop the third point, salt water and freshwater meet in coastal areas which create brackish water. Because the salt water and the brackish water are heavier than freshwater, they stay in the bottom of the aquifer.

However with the pumping wells, brackish water is sucked into the wells and then rises to a higher level than it should be. It can even reach the soil sometimes which makes agriculture impossible.

The problem of the (very) slow natural remediation does not only concern the first point. Indeed, removing freshwater increases the level of brackish water in the aquifer. Then brackish water can then enter in the pumping well. Some solutions exist to this problem but they are most of the time technically impossible and/or way too expensive.

For example it is shown in the talk that we could build a barrier to separate salt water and freshwater but it is almost impossible to do for technical and financial issues. There is the same problem for the second option which consists to create a new coastal land to increase the freshwater volume.

In the end, the only possible way to fight against salinization and the lack of freshwater is to put some freshwater in the aquifer to make the salt water fall back because of its bigger density and help the aquifer to recover. Of course this opreation is very expansive but remains possible which is not the case for the two other solutions.

1.3 A pragmatic question

As seen in the previous part, there is no good way for us to manage the aquifers. That is why we need to create tools to understand and control them better.

First, we look at why this problem is not interesting in fluid dynamics. There are mainly 3 reasons to this :

- First, the water in the aquifer is not dynamic horizontally (1 to 4 km/year). This property brings us an approximation to find the height of the interface between the freshwater and salt water. However we cannot use it because the error is of the order of the meter which is actually the depth of our aquifer
- Then, this kind of problem have already been mainly studied because of oil extraction

- Finally, these models are too sensitive to parameters which we do not necessarily have access or at least can measure precisely enough

Then we look at an optimal control problem to illustrate the sensitivity of the fluid dynamic model to the parameters. This problem is to maximise the income of the farmer who has an aquifer under his crops which implies he has to pollute the aquifer with some fertilizer while considering the cleaning costs that it will generate.

We can prove that this optimal control problem got a unique solution when the fertilizer concentration remains small. This implies we can make some simulations on it. Indeed, these simulations shows us that even a slight variation of the boundary conditions completely changes the optimal politics and this phenomenon is even more observable when we do not make the assumption that the fluid is quasi-stationnary.

Imagine now if we consider some parameters that we cannot control at all, for example the rain. It makes this model completely useless (even if assuming that the fluid is quasi-static was already a big assumption).

However, this problem can be an interesting dynamic fluids one.

Indeed, the groundwater flow is slow (goes horizontally between 1 and 4 km/year) and the recharge time is about 1-30 hours which is very quick in our case. It implies that we cannot hope to find a time scaling for such different phenomenons, we will then have to find a new way to define our solution.

As said before, we have already studied these models with oil extraction. It means that what we still not have demonstrated would be groundbreaking.

Finally, the fact that we do not have access to most of the parameters makes us move from applied mathematics to applicable mathematics.

2 Interaction between ground and surface water

2.1 The "right" model

We first look at the 3D Richards equation when assuming that the pressure of air is equal to the atmospheric pressure (realistic because of gravity) :

$$\partial_t s + div(v) = 0$$

with s and v both depending on the pressure

The problem with this model is that it is degenerated in time and in space.

As said before, considering our problem as a fluid flow in a partially porous medium problem brings us a risk to lose the unknown.

That's the degeneration we find in the 3D Richards equation : when the pressure is too high, the aquifer is completely saturated which means that saturation is equal to one. In this case, the term $\partial_t s$ disapears. Then we lose the time information.

The space derivative term can also disapear when the medium is completely dry which means that the permeability equals zero. The equation to define v is the following:

$$v = -k_r(P)K_0(\frac{1}{\rho g}\nabla P + e_3) \qquad \text{in}(0,T)\times\Omega$$

It means that when the permeability K_0 goes to zero, v also equals zero and thus $div(v)$ equals zero.

In this case we lose the space information.

2.2 Basic idea for a simplified model

The problem with this model is its computation time which is way to long and gives non-negligible errors.

Therefore, we can split this 3D equation into 2 equations, the first one is a 1D vertical Richards equation and the second one is a 2D simpler model (for example a stationnary model).

The reason why we can do this is that because of gravity, the water displacement in the upper part (above the water table) is mainly vertical and pretty quick. Under the water table, the displacement is mainly horizontal and way slower which let us do some approximation and thus, simplify the model.

The biggest problem when we split the 3D Richards equation is how we define the water table because it has to be an unknown of the problem. Indeed, when water goes in the aquifer, the saturated zone gets higher and so does the water table, we have to consider it.

We also have to look at the error of this model, winning time computation to lose in precision is not great for us.

This model has already been studied but it was not mass conservative which is important in our case because the water that goes out the upper part goes in the lower one.

2.3 Usual approach for the justification of a simplified model

First, we look how to valid a model and the usual way to do it is through asymptotic analysis.

To do this we will introduce a term which depends on caracteristics of our model and make it tends to zero. As said before, the depth of our aquifers (under 8m) is way smaller than their lenght (in the order of the kilometer) which brings us the following approximation in a shallow aquifer :

We can define epsilon as the sqaure of the rate between the depth and the lenght of our aquifer and make it tends to zero :

$$\epsilon = \frac{D^2}{L^2}$$

With this we are able to only look at the main order terms and do our asymptotic analysis.

We also have to chose a caracteristic time and in this case the chosen one is huge which is a problem for us because the natural remediation only take hours.

We will then try to construct a model matching the asymptotic behavior which is valid for any scale of time.

2.4 A new class of models

We now introduce a new model which couples a vertical 1D Richards equation for the unsaturated part of our aquifer and a horizontal 2D stationnary Richards equation for the saturated part. We also want this model to be valid at any time scale which implies different equations for the different scales.

The 1D vertical Richards equation is the following :

$$\begin{cases} \partial_t s(P) + div(u.e_3) = 0 \qquad \text{for } t \in \,]0,T[\,, (x,z) \in \Omega_h^+(t) \\ \alpha P + \beta u.e_3 = F \qquad \text{for } (t,x) \in \,]0,T[\, \times \Gamma_{soil} \\ P(t,x,h(t,x)) = \rho g(\tilde{H}(t,x) - h(t,x)) \qquad \text{for } (t,x) \in \,]0,T[\, \times \Omega_x \\ P(0,x,z) = P_{init}(x,z) \qquad \text{for} (x,z) \in \Omega_h^+(0) \end{cases}$$

with T our caracteristic time, $\Omega_h^+(t)$ the upper part of our aquifer (above our frontier $h(t)$), Γ_{soil} the ground level and Ω_x the lower part of our aquifer.

We can underline that our frontier h depends on time because it's also an unknown of our problem.

I will explain what I understood on every equation :

- The first one is the same as the 3D Richards equation written before but the componnent in space and then the space derivative is only in 1D.

  Indeed, u is the horizontal componnent of the velocity of our fluid and e_3 is the horizontal basis vector in space. Time t is taken is our observation interval and we also define x and z in the upper part $\Omega_h^+(t)$ because the pressure P is a function of x and z
- The second one is a boundary condition at the ground level which is the upper boundary of the aquifer's aeration part. The pressure and the vertical component are both taken into account because they define what comes from the soil and goes in the upper part. What comes from the soil (rain etc.) is defined with a function F which gives the second equation. Two coefficients α and β are added and are define by measures to create a realistic model
- The third one is a Dirichlet boundary condition beacause we set the value of pressure at the interface level. It is a function of the density of the medium ρ, of gravity g and of the pressure's gradient
- The last one simply initializes P at time $t = 0$

The 2D horizontal Richards is the following :

$$\begin{cases} div_x(\tilde{K}(\tilde{H})\nabla_x \tilde{H}) = (u.e_3)|_{\Gamma_h^+} \qquad \text{for } (t,x) \in \,]0,T[\, \times \Omega_x \\ \tilde{K}(\tilde{H})\nabla_x \tilde{H}.n = 0 \qquad \text{for } (t,x) \in \,]0,T[\, \times \partial\Omega_x \\ \tilde{H}(0,x) = H_{init}(x) \qquad \text{for } x \in \Omega_x \end{cases}$$

- The first equation defines what goes in the lower part of the aquifer from the upper part term $(u.e_3)$ taken at the frontier h

- The second one is a Neumann boundary condition to modelize the fact that water in the saturated part cannot come back in the upper part or enter deeper in the ground (represented in Figure 1 by the "very low hydraulic-conductivity bedrock")
- The last equation initializes $\tilde{H}$ at time $t = 0$

We then have to look at the asymptotic analysis to valid this model. We derive the asymptotic models for every time scale, we link them with corrective terms and we finally get something to compare with the 3D Richards equation. To do it, we introduce the same epsilon we defined before and we make it tends to zero for every time scale.

The result we get is that the two models (the Richards and the coupled one) have the same asymptotic behavior which is what we expected.

The coupling gives us the mass conservation, we created the model in this goal (the existing models did not give this property) so we can check the mass conservation very easily.

Indeed, the source term for the lower part is the mass from the upper part.

We here talk about class of models because the frontier between the upper and the lower part changes. The way we define this separation brings us a new model each time.

To determine how we should define our frontier, we compare 3 types of separations to the 3D Richards model. In the end, we see that considering the frontier at the level of the water table is not the best choice even if it seemed very natural.

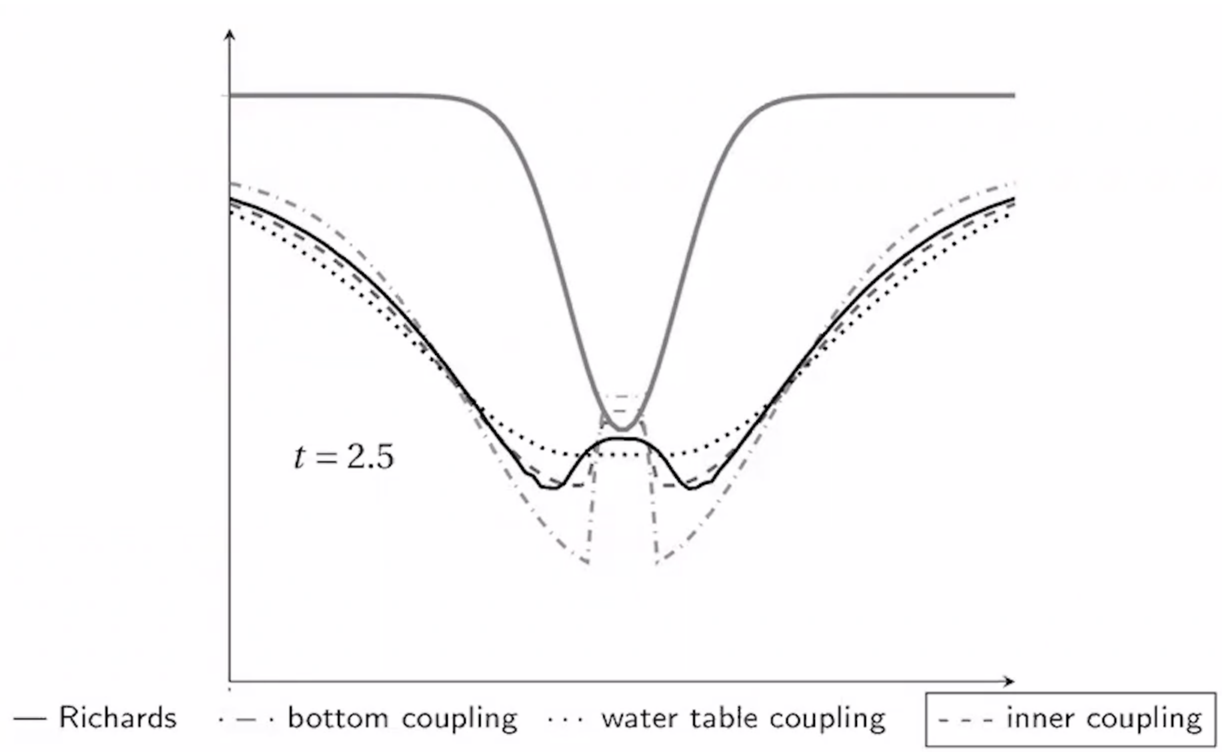

Figure 3 : Different couplings simulated with 3D Richard equation

The reason why it does not match if we do a bottom coupling is the following :

The vertical component is too big in the lower part to be negligible and the horizontal one is too small to handle saturation inferface which makes our simulated interface under the real one.

For the water table coupling it is the converse default, the vertical component is too small and the horizontal one too big which gives us a simulated inferface above the real one.

That is the reason why the inner coupling is taken.

I want to add that the inner coupling choice was not proved enough in the talk because we actually see that it does not work for the two other coupling but nothing says it work for the inner one. We can only rely on the simulations here.

As we said, we constructed a class of models so we can define our inner coupling in infinite ways. Thus, we will use the deeplearning to find the best definition for our frontier.

3 Theoretical analysis

By looking at the state of the art, we can hope to find a weak solution at our problem.

First, we couple the two set of equations and we apply a test function ϕ to find a weak solution to our problem. Doing this simplifies the calculations but brings some troubles too.

For example we go back to a 3D problem which is bad beacause this model was actually built to get rid of 3D and accelerate the computation. We also extend the domain where the equation degenerates to the whole space which is a pretty bad new.

Then we try to get another definition which preserves the dimension reduction. We uncouple the problem and define the interface so it is not a function of time anymore and try to apply a fixed point strategy. With this, the Dirichlet boundary condition in the 1D Richards equation also does not depends on time anymore.

The resulting equation brings some new hypothesis for the solution :

First, we cannot weaken too much the solution because we will then lose compactness results. We also have now a time derivative term in h_{bc} which is our new interface. It forces $h_{bc}(., x) \in H^1(0,T)$ to keep some control on this term.

Then we look if the application $h_{bc} \mapsto \tilde{h}$ admits a fixed point. The problem is that $h_{bc}(., x) \in H^1(0,T)$ and the equation we get does not depend on time. Thus, we add a time derivative term on $\tilde{h}$ to solve this inconvinence.

In the end, we get two different definitions for weak solutions :

- The first one loses the dimension reduction, we went back to a 3D problem
- The other one loses the stationnary caracteristic

Some issues are raised by this result :

Should we consider those simplified models unsuitable for mathematical analysis or should we develop new tools to improve them ? Ms.Choquet clearly thinks we should find new tools and it is exciting !

4 Self-improving properties

In this last part we will look at the possible impromments for the models and their weak solution.

For example if we go back to a linear setting get some regularity on the source term, we can obtain greater properties for the regularity of the solution with Meyers' results. Even if the gain could be zero in theoretical cases, it seems to be interesting on real applications.

However, it has been proved that we cannot get as much regularity as we want.

In the end we see that the norm of the inverse of the heat operator can bring us results. It let some hope and ongoing work.

5 Conclusion

We used the properties of unconfinned aquifers (shallow character and two separated zones) to create a viable model at any scale of time and faster to compute.

To reduce the computation time we splitted the original 3D model into a vertical 1D model and a horizontal 2D model.

We coupled them at the interface to conserve mass.

However, the interface is also an unknown of the problem so it created a whole class of models.

We used deeplearning to find the best coupling which occurs to be an inner one.

In the end, we defined two types of weak solutions to our model :

- The first one lost the dimension reduction property and degenerates in the whole space
- The second one loses the stationnary property

It shows us that we still have improvements to do on our model.